Length of separable states and symmetrical informationally complete (SIC) POVM

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This short note reviews the notion and fundamental properties of SIC-POVM and its connection with the length of separable states. We also review the t-design.

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I. DEFINITION AND BACKGROUND OF SIC-POVM

1. [3, 12, 17] In the d-dimensional Hilbert space, a SIC-POVM consists of d^2 outcomes that are subnormalized projectors onto pure states $\Pi_j = \frac{1}{d}|\psi_j\rangle\langle\psi_j|$ for $j, k = 1, \dots, d^2$, such that

$$|\langle\psi_j|\psi_k\rangle|^2 = \frac{1 + d\delta_{jk}}{d+1}. \tag{1}$$

2. [12, Theorem 2] Using Eq. (1) we can show that any SIC-POVM forms a 2-design:

$$\sum_{i=1}^{d^2} |\psi_i, \psi_i\rangle\langle\psi_i, \psi_i| = \frac{2d}{d+1} S_d. \tag{2}$$

   Here, the operator S_d denotes the $d \times d$ symmetrizer operator, i.e.,

$$S_d := \sum_{i=1}^{d} |ii\rangle\langle ii| + \sum_{j>i=1}^{d} \frac{|ij\rangle + |ji\rangle}{\sqrt{2}} \frac{\langle ij| + \langle ji|}{\sqrt{2}}. \tag{3}$$

3. Eq. (2) implies that $\sum_{j=1}^{d^2} \Pi_j = I$, so SIC-POVM is a complete measurement in physics.

4. Three basic papers on SIC-POVMs are [3, 12, 17].

   (1) G. Zauner, "Quantendesigns - Grundzuge einer nicht kommutativen Designtheorie," PhD thesis (University of Vienna, 1999).

   (2) J. M. Renes, R. Blume-Kohout, A. J. Scott, and C. M. Caves, J. Math. Phys. 45, 2171 (2004). (provide analytical $d = 2, 3, 4$, numerical $d \le 45$.)

   (3) D. M. Appleby, J. Math. Phys. 46, 052107 (2005). It provides the analytical solutions of SIC-POVM for $d = 2, \cdots, 7, 19$.

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5. **Example 1** *SIC-POVM for $d = 2$. Let*

$$
\begin{aligned}
|\psi_0\rangle &= \sqrt{\frac{3+\sqrt{3}}{6}}|0\rangle + e^{\pi i/4}\sqrt{\frac{3-\sqrt{3}}{6}}|1\rangle, &(4)\\
|\psi_1\rangle &= \sqrt{\frac{3+\sqrt{3}}{6}}|0\rangle - e^{\pi i/4}\sqrt{\frac{3-\sqrt{3}}{6}}|1\rangle, &(5)\\
|\psi_2\rangle &= \sqrt{\frac{3+\sqrt{3}}{6}}|1\rangle + e^{\pi i/4}\sqrt{\frac{3-\sqrt{3}}{6}}|0\rangle, &(6)\\
|\psi_3\rangle &= \sqrt{\frac{3+\sqrt{3}}{6}}|1\rangle - e^{\pi i/4}\sqrt{\frac{3-\sqrt{3}}{6}}|0\rangle, &(7)
\end{aligned}
$$

. Then one can verify

$$\sum_{i=0}^{3} |\psi_i, \psi_i\rangle\langle\psi_i, \psi_i| = \frac{4}{3} S_2. \tag{8}$$

The four states $|\psi_i\rangle, i = 1, 2, 3, 4$ form a regular tetrahedron when represented on the Bloch sphere.

6. Analytical SIC-POVMs have been constructed for dimension $d = 2, \cdots, 16, 19, 24, 28, 31, 35, 37, 43, 48$, see [14]. Numerical SIC-POVMs have been constructed for $d \le 67$, see the details in [20]. This is achieved by the popular method of Weyl-Heisenberg group in quantum information community. However the construction becomes hard for higher dimensions. So it is unknown, though widely believed, that whether SIC-POVM exists for any dimension d.

7. Constructing SIC-POVM is one of the most important questions in quantum information. It is related to quantum tomography [18], Mutually unbiased bases (MUBs) [4, 16], entanglement theory [7, 19], Lie Algebra [2], Galois field [1], foundations of quantum mechanics [9] and so on.

II. RELATING SIC-POVM TO LENGTH

For a bipartite state ρ acting on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, the partial transpose computed in the standard orthonormal (o.n.) basis $\{|i\rangle\}$ of system A, is defined by $\rho^\Gamma = \sum_{ij} |j\rangle\langle i| \otimes \langle i|\rho|j\rangle$. One can similarly define the partial transpose Γ_B on the system B. Let $r(\rho)$ denote the rank of ρ. We call the integer pair $(r(\rho), r(\rho^\Gamma))$ the *birank* of ρ, and the two integers may be different. The *length*, $L(\rho)$, of a separable state ρ is the minimal number of pure product states over all such decompositions of ρ [8]. It is known that $L(\rho) \ge \max\{r(\rho), r(\rho^\Gamma)\}$.

One can verify that the partial transpose of the state $\rho_2 = \frac{2}{d^2+d} S_d$ is

$$\rho_2^\Gamma = \frac{1}{d^2+d}(I + |\Psi_d\rangle\langle\Psi_d|), \tag{9}$$

where $|\Psi_d\rangle = \sum_{i=1}^d |ii\rangle$ is the non-normalized d-level maximally entangled state. So the separable state ρ_2 has birank $(\frac{d^2+d}{2}, d^2)$. Therefore we have $L(\rho_2) \ge d^2$. The equality holds for $d = 2$ by Example 1. It also holds for $d = 2, \cdots, 16, 19, 24, 28, 31, 35, 37, 43, 48$ [14]. However the question is whether

Conjecture 2 $L(\rho_2) = d^2$ *for any* $d \ge 2$.

The positive answer of this conjecture would imply that the SIC-POVM exists for any integer $d \ge 2$. This argument has been proved by using the notion of weighted 2-design in [13, Theorem 4]. On the other hand if Conjecture 2 turned out to fail for some d, i.e., $L(\rho_2) > d^2$, then SIC-POVM would not exist for this d. This argument has been proved by Eq. (2) and [12, Theorem 2].

To conclude, either the positive or negative answer to Conjecture 2 will solve the SIC-POVM problem.

III. MORE GENERAL BACKGROUND: T-DESIGN

Let $t \geq 1$ be an integer. The t-design of dimension d is defined as a set S of pure product states $|a_i\rangle \in \mathbf{C}^d$ if

$$\frac{1}{|S|} \sum_i |a_i\rangle\langle a_i|^{\otimes t} = \rho_t = \binom{d+t-1}{t}^{-1} S_{d,t}, \tag{10}$$

where $S_{d,t}$ is the t-partite symmetrizer operator in the space $(\mathbf{C}^d)^{\otimes t}$. For example, $S_{d,t} = S_d$ for $t = 2$ in Eq. (3). It is known [5, 13] that the number of design points satisfies

$$|S| \geq \binom{d + \lfloor t/2 \rfloor - 1}{\lfloor t/2 \rfloor} \binom{d + \lceil t/2 \rceil - 1}{\lceil t/2 \rceil}. \tag{11}$$

A design which achieves this lower bound is called *tight*. For example, the bound is equal to d, d^2 and $d^2(d+1)/2$ for $t = 1, 2, 3$, respectively. The t-designs exist for any d [15]. In the language of quantum information, it means that any t-partite symmetrizer operator is a non-normalized separable state. However it is unknown that whether tight t-designs exist, i.e., whether the length of t-partite symmetrizer operator reaches the lower bound in Eq. (11).

Here are a few known results from the field of t-designs. For $d = 2$, tight t-designs exist for $t = 1, 2, 3, 5$ [11]. For a few $d > 2$, tight t-designs exist for $t = 1, 2, 3$ [5, 6]. Here is the detail. It is trivial that tight 1-designs exist for any d. The existence of tight 2-designs is equivalent to the positive answer for Conjecture 2, in terms of Eq. (10). So far this is true for $d = 2, \cdots, 16, 19, 24, 28, 31, 35, 37, 43, 48$, see [14]. Third, the tight 3-designs are known only for $d = 2, 4, 6$ [10]. In particular for $d = 2$, the six states from an MUB in $\mathbf{C}^2$ form a tight 3-design [20]. It can also be directly verified by computing the frame potential.

Note that ρ_t is a t-partite separable state. We have

Lemma 3 *The tight t-design of dimension d exists if and only if* $L(\rho_t) = \binom{d+\lfloor t/2 \rfloor - 1}{\lfloor t/2 \rfloor} \binom{d+\lceil t/2 \rceil - 1}{\lceil t/2 \rceil}$.

The proof is based on Ref. [41,42] of [13]. Nevertheless, it is known that the tight t-design does not exist for $d \geq 3, t \geq 5$ [13].

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